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Third-order approximation to short-crested waves

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Short-crested wave systems, as produced by two progressive waves propagating at an oblique angle to each other, have an extremely important effect on a sedimentary bed. The complex water-particle motions are conducive to lifting material into suspension and sustaining it in motion. In order to study this phenomenon rigorously, the variables of this wave system are derived to a third-order approximation by a perturbation method. The case of waves reflecting obliquely from a vertical wall is examined under the assumptions of full reflexion, uniform finite depth and an inviscid incompressible fluid. The new formulation reduces to standing or Stokes waves at the limiting angles of approach. Expressions for kinematic quantities are also presented.

1. Introduction

Wave reflexion from maritime structures, particularly seawalls and jetties, has been extensively examined both theoretically and experimentally for the two-dimensional case. It is relatively easy to produce the kinematic characteristics of standing waves to higher order and then to verify them in flume tests. However the case of oblique reflexion, resulting in short-crested wave systems, has received very little attention.

This omission is understandable owing to the complexity of the phenomenon, but may also be due to the non-recognition of its wide occurrence in nature and its engineering importance. The production of a complete standing wave or *clapotis* in the laboratory is always difficult because of the implied equality of the amplitude and frequency of the incident and reflected wave. Its occurrence in nature, therefore, is even less likely since complete reflexion and the presence of secondary waves will continually detract from the ideal form. But the major difference in natural conditions is the changeability of the incident wave direction.

Any oblique approach to a wall alters the kinematics of the water particles drastically from the two-dimensional model (Hsu 1977). Short-crested waves or *clapotis gaufré* occur in conditions other than simple reflexion, for example through diffraction behind an offshore structure or island, differential reflexion of swell waves of differing period across the continental shelf or shoal, concurrent arrival of swell waves from different storm zones, and even in the generation process itself. Thus it could be said that for engineering applications the propagation of two wave trains at an angle to each other is of greater importance than that of progressive waves with a single direction, even when a spectrum may be included in the analysis or experiment.

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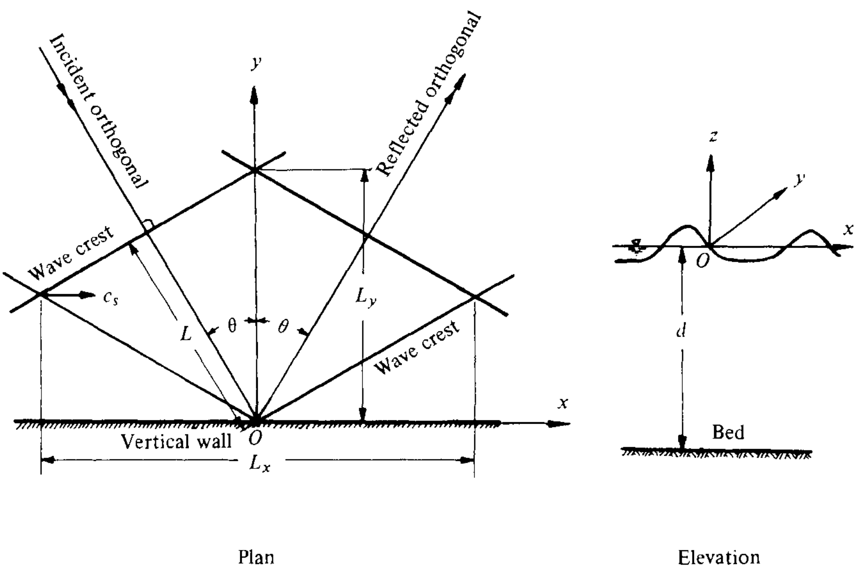

FIGURE 1. Definition sketch of short-crested waves produced by reflexion showing co-ordinates.

The computation of wave forces on structures should assume the worst conditions, which result from pure standing waves. On the other hand, the scouring from wave action at a sedimentary bed must take account of oblique reflexion. Silvester (1972) has reported tests on a movable bed model involving short-crested waves in front of a wall. He has noted a possible application in the transmission of littoral drift across river or harbour mouths without bar formation (Silvester 1975). He has also pointed out the importance of submerged reefs in the case of coastal sediment (Silvester 1977). In all such cases it is the persistent swell, which is virtually a single incident wave train, that produces the long-term scour and not the multi-directional and multi-train storm waves. The latter could well cause a catastrophe whose origin was the long-period low-amplitude swell arriving continually from distant storms.

In a long-term research programme into the effects of short-crested waves on sediment movement it is first necessary to know the kinematics of the water-particle motion. With this knowledge the placement of sand grains into suspension could be predicted for any incident wave in a particular depth of water. The horizontal movement of this material could then be related to the mass transport of the water particles due to this complex wave system. Such net motion will vary across the crest length, or normal to the propagation direction of the combined crests, being maximal along the crests and minimal half-way between.

Short-crested waves are defined as having a surface elevation which is doubly periodic in two perpendicular directions, these being along the reflecting wall and normal to it. A symmetrical diamond-shaped crest pattern, with flat troughs and steep crests, can represent the short-crested wave system in plan as seen in figure 1. Since most engineering applications will be for finite depths of water (not necessarily shallow in the mathematical sense) and for waves of finite amplitude, a general solution is therefore required which is a higher approximation than linear, and of at least third order.

Fuchs (1952) obtained a second-order solution, based upon the work of Stokes (1847), but did not test it to the limit of a standing wave in the case of normal approach to the wall. Chappelear (1961) extended this to third order in the same manner using a formal power expansion. Both of these solutions are in dimensional form, assuming steady motion when viewed from a co-ordinate system moving parallel to the wall with the speed of the wave.

Chappelear (1961) employed an expansion parameter which was proportional to the ratio of wave height to wavelength of the short-crested system measured parallel to the wall. This is similar to the derivation of progressive waves of finite amplitude in that direction, but precludes the possibility of obtaining the standing wave, which is two-dimensional and at right angles to the wall, in the limit of normal incident approach.

To overcome this difficulty the authors have related the wave steepness parameter to the wavelengths of the incident and reflected waves, which in the present case are assumed to be the same. This permits the solution to encompass all angles of incidence and thus the solution can be extended to the limits for both standing and Stokes waves. Another innovation, previously employed by Tadjbakhsh & Keller (1960) in deriving standing waves of finite amplitude, is used in the simplification of the analysis. This is the dimensionless angular wave frequency [see equation (23)], which makes the complicated series involved in this higher-order solution simpler to write down.

The assumptions employed are an inviscid incompressible fluid of uniform and finite depth. Also complete reflexion has been assumed, which implies equal wave height and period of the reflected and incident wave. The solution cannot, therefore, be used for short-crested systems produced by component waves of differing period or height, or the partial standing wave in the limit. For the case of waves reflecting obliquely from a vertical wall, the Mach-stem effect has been omitted. There are, however, many important engineering applications for such complex solutions if and when they become available.

The general solutions for the velocity potential and surface elevation obtained for each order of approximation are finally rearranged to give some of the more important physical quantities of this wave system. These include the surface elevation, the crest height above the still water level, the wave speed and the Eulerian water-particle velocities, plus the wave pressure variation throughout the whole regime. All solutions are presented in series form.

2. Perturbation method

Exact solutions for nonlinear problems in continuum mechanics are rare. A solution for the short-crested wave system is therefore obtained by successive approximations. A perturbation procedure is employed to solve the nonlinear governing equations for three-dimensional irrotational water waves in a finite depth. The set of linear equations so deduced then yields the desired solution to the original problem.

Consider, as in figure 1, a three-dimensional irrotational wave motion bounded above by a free surface, below by a rigid horizontal bed and to one side by a rigid vertical wall. Assume the fluid to be inviscid, incompressible and of uniform finite depth. The resulting diamond-shaped crest pattern due to oblique reflexion will assume full wave reflexion, which implies equal amplitudes and periods for incident and reflected components. The Mach-stem effect, which occurs at large angles between the incident wave crest and the wall, will be ignored.

Using Cartesian co-ordinates as in figure 1, the velocity potential ϕ for irrotational flow gives the Eulerian velocity components of water particles as

$$u = \phi_x, \quad v = \phi_y, \quad w = \phi_z, \tag{1}$$

where the velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \tag{2}$$

The dynamic and kinematic boundary conditions at the free surface are

$$\phi_t + g\eta + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0, \tag{3}$$

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0, \tag{4}$$

while the bottom and wall boundary conditions are

$$\phi_z = 0 \quad \text{at} \quad z = -d, \quad \phi_y = 0 \quad \text{at} \quad y = 0, \tag{5}$$

in which η is the water surface elevation, g the acceleration due to gravity, x, y and z the Cartesian co-ordinates and t the time.

Putting all equations into dimensionless form, let ϵ represent the small parameter ka, where a is the amplitude of the short-crested wave to first order and k is the wavenumber $2\pi/L$, L being the wavelength of the incident or reflected wave. The following dimensionless quantities may then be introduced:

$$\left.\begin{aligned} &\hat{x} = kx, \quad \hat{y} = ky, \quad \hat{z} = kz, \quad \hat{t} = \sigma t, \quad \hat{d} = kd, \\ &\hat{\phi}(\hat{x},\hat{y},\hat{z},\hat{t}) = \frac{k^2}{\epsilon(gk)^{\frac{1}{2}}}\,\phi(x,y,z,t), \quad \text{potential function,} \\ &\hat{\eta}(\hat{x},\hat{y},\hat{t}) = (k/\epsilon)\,\eta(x,y,t), \quad \text{surface elevation,} \\ &\omega = \sigma/(gk)^{\frac{1}{2}}, \quad \text{angular frequency,} \end{aligned}\right\} \tag{6}$$

in which σ is the angular frequency of the incident (and reflected) wave (i.e. $2\pi/T$, where T is the wave period in seconds).

The carets denoting dimensionless quantities will now be omitted for the sake of simplicity, unless otherwise specified. The governing equations (2)–(5) may now be transformed in terms of these dimensionless quantities:

$$\nabla^2\phi = 0, \tag{7}$$

$$\eta + \omega\phi_t + \tfrac{1}{2}\epsilon(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad \text{at} \quad z = \epsilon\eta, \tag{8}$$

$$\phi_z - \omega\eta_t - \epsilon\eta_x\phi_x - \epsilon\eta_y\phi_y = 0 \quad \text{at} \quad z = \epsilon\eta, \tag{9}$$

$$\phi_z = 0, \quad \text{at} \quad z = -d, \quad \phi_y = 0 \quad \text{at} \quad y = 0. \tag{10}$$

In order to solve for the current unknowns (ϕ, η and ω) the short-crested wave must be specified. As shown in figure 1, let L be the wavelength of the incident and reflected waves and L_x and L_y the distances between crests in the x and y directions. Then the components of the wavenumber k may be defined respectively as

$$m_1 = 2\pi/L_x = k\sin\theta = mk, \quad n_1 = 2\pi/L_y = k\cos\theta = nk, \tag{11}$$

where θ is defined as in figure 1; thus

$$m_1^2+n_1^2 = k^2, \quad m^2+n^2 = 1. \tag{12}$$

When θ becomes zero, a normal standing wave occurs; when it is $\frac{1}{2}\pi$, a progressive wave equal to the incident component results.†

To determine some coefficients of the general solution, additional conditions for $\theta = 0$ and $\frac{1}{2}\pi$ are required. For the complete standing wave, conditions which were specified by Tadjbakhsh & Keller (1960) are used with some necessary modifications as follows:

$$\int_0^{\pi}\int_0^{\pi} \eta(x,y,t)\,dy\,dx = 0, \tag{13}$$

$$\nabla\phi(x,y,z,t+2\pi) = \nabla\phi(x,y,z,t), \tag{14}$$

$$\int_0^{\pi}\int_0^{2\pi} \eta(y,t)\sin t\cos y\; dt\,dy = 0, \tag{15}$$

$$\int_{-d}^{0}\int_0^{\pi}\int_0^{2\pi} \phi(y,z,t)\sin t\,\cos y\; dt\,dy\,dz = \tfrac{1}{2}\pi^2(\tanh d)^{\frac{1}{2}}. \tag{16}$$

Equation (13) applies to the conservation of water mass, whilst (14) determines the periodicity of the wave motion, both extended to the three-dimensional case. Equations (15) and (16) dictate the phase and amplitude of wave motion.

The progressive wave must be specified by

$$\int_0^{2\pi} \eta(x,t)\,dx = 0, \tag{17}$$

$$\nabla\phi(x,z,t+2\pi) = \nabla\phi(x+2\pi,z,t) = \nabla\phi(x,z,t), \tag{18}$$

which preserve the water mass, as in (13), and the periodicity in time and space.

To provide a unique solution a further condition, introduced first by Tadjbakhsh & Keller (1960), must be imposed, namely

$$n'\tanh n'd/\tanh d \neq j^2 \quad \text{for} \quad n' \geqslant 2, \quad j \geqslant 1. \tag{19}$$

A solution is sought for any value of d, but it will not be unique in satisfying (7)–(10) unless it also satisfies (19). A discussion of this condition is included in the appendix.

The problem set is that of determining solutions for the dimensionless physical quantities ϕ, η and ω that satisfy (7)–(10) and the additional conditions mentioned. It is assumed that these variables can be expanded as power series in the small parameter ϵ as

$$\left.\begin{aligned} \phi(x,y,z,t) &= \phi_0+\epsilon\phi_1+\tfrac{1}{2}\epsilon^2\phi_2+\ldots, \\ \eta(x,y,t) &= \eta_0+\epsilon\eta_1+\tfrac{1}{2}\epsilon^2\eta_2+\ldots, \\ \omega &= \omega_0+\epsilon\omega_1+\tfrac{1}{2}\epsilon^2\omega_2+\ldots. \end{aligned}\right\} \tag{20}$$

† The relationships $\hat{x} = m_1 x$ and $\hat{y} = n_1 y$ were tried first, instead of $\hat{x} = kx$ and $\hat{y} = ky$. However, it was found that these could not separate completely the x from the y components in the final equations for the two-dimensional cases.

The dimensionless velocity potential at the free surface may be expressed in terms of the Taylor expansion at $z = 0$ instead of $z = \epsilon\eta$, so that

$$\begin{aligned}\phi(x,y,\epsilon\eta,t) = \phi_0 + \epsilon(\eta_0\phi_{0z}+\phi_1) + \epsilon^2(\eta_1\phi_{0z}+\tfrac{1}{2}\eta_0^2\phi_{0zz}+\eta_0\phi_{1z}+\tfrac{1}{2}\phi_2) \\ +\epsilon^3(\tfrac{1}{2}\eta_2\phi_{0z}+\eta_0\eta_1\phi_{0zz}+\tfrac{1}{6}\eta_0^3\phi_{0zzz}+\eta_1\phi_{1z}+\tfrac{1}{2}\eta_0^2\phi_{1zz} \\ +\tfrac{1}{2}\eta_0\phi_{2z}+\tfrac{1}{6}\phi_3)+O(\epsilon^4). \qquad (21)\end{aligned}$$

Substituting (21) into (8) and (9) and collecting terms of each order in ϵ yields the necessary equations to each order of approximation for conditions at $z = 0$.

3. Third-order approximation

A generalized three-dimensional wave theory for the short-crested system will now be formulated to third order. The two expanded free-surface boundary conditions (8) and (9) will be used to obtain linear partial differential equations for each order of approximation in terms of ϵ. The consistent form of $\cos ny \sin(mx-t)$ will be adopted in formulating expressions for all potential functions for convenience of direct comparison with the Stokes wave (Skjelbreia 1959) and the standing wave (Goda & Kakizaki 1966) in the limiting conditions. The governing equations for each order of approximation are listed together with the procedure for solving them. The resulting expressions for the velocity potential, surface elevation and angular frequency are presented. The limiting two-dimensional cases are discussed.

First-order approximation

On introducing (20) and (21) into (7)–(10), the terms in ϵ^0 are given by

$$\left.\begin{aligned}&\phi_{0xx}+\phi_{0yy}+\phi_{0zz}=0,\\ &\eta_0+\omega_0\phi_{0t}=0, \quad \phi_{0z}-\omega_0\eta_{0t}=0 \quad \text{at} \quad z=0,\\ &\phi_{0z}=0 \quad \text{at} \quad z=-d, \quad \phi_{0y}=0 \quad \text{at} \quad y=0.\end{aligned}\right\} \qquad (22)$$

The solutions which satisfy the boundary conditions are easily obtained in a dimensionless form similar to that given by Tadjbakhsh & Keller (1960) and Goda & Kakizaki (1966) as follows:

$$\left.\begin{aligned}&\phi_0=\omega_0\frac{\cosh(z+d)}{\sinh d}\cos ny\sin(mx-t),\\ &\eta_0=\cos ny\cos(mx-t), \quad \omega_0^2=\tanh d,\end{aligned}\right\} \qquad (23)$$

in which m and n are defined as in (11). Equation (23) yields the appropriate limiting cases of a progressive wave as $n \to 0$ and a standing wave as $m \to 0$.

Second-order approximation

In the same manner as for the first-order approximation, the terms in ϵ are given by

$$\left.\begin{aligned}&\phi_{1xx}+\phi_{1yy}+\phi_{1zz}=0,\\ &\eta_1+\omega_0\phi_{1t}=-\omega_1\phi_{0t}-\omega_0\eta_0\phi_{0zt}-\tfrac{1}{2}(\phi_{0x}^2+\phi_{0y}^2+\phi_{0z}^2) \quad \text{at} \quad z=0,\\ &\phi_{1z}-\omega_0\eta_{1t}=\omega_1\eta_{0t}+\eta_{0x}\phi_{0x}+\eta_{0y}\phi_{0y}-\eta_0\phi_{0zz} \quad \text{at} \quad z=0,\\ &\phi_{1z}=0 \quad \text{at} \quad z=-d, \quad \phi_{1y}=0 \quad \text{at} \quad y=0.\end{aligned}\right\} \qquad (24)$$

By substituting (23) into the right-hand side of (24), the dynamic free-surface condition is transformed to

$$\eta_1 + \omega_0 \phi_{1t} = \omega_1 \omega_0^{-1} \cos ny \cos(mx-t) + \tfrac{1}{8}[3\omega_0^2 - \omega_0^{-2}(m^2+n^2)\cos 2ny \cos 2(mx-t)]$$
$$+ \tfrac{1}{8}[3\omega_0^2 - \omega_0^{-2}(m^2-n^2)]\cos 2(mx-t)$$
$$+ \tfrac{1}{8}[\omega_0^2 - \omega_0^{-2}(m^2-n^2)]\cos 2ny + \tfrac{1}{8}[\omega_0^2 - \omega_0^{-2}(m^2+n^2)] \quad \text{at} \quad z=0 \tag{25}$$

while the kinematic free-surface condition becomes

$$\phi_{1z} - \omega_0 \eta_{1t} = \omega_1 \cos ny \sin(mx-t) - \tfrac{1}{4}\omega_0^{-1}(m^2+n^2+1)\cos 2ny \sin 2(mx-t)$$
$$- \tfrac{1}{4}\omega_0^{-1}(m^2-n^2+1)\sin 2(mx-t) \quad \text{at} \quad z=0. \tag{26}$$

By differentiating (25) with respect to t and eliminating η_{1t} from (25) and (26), the combined free-surface boundary condition is given by

$$\phi_{1z} + \omega_0^2 \phi_{1tt} = 2\omega_1 \cos ny \sin(mx-t) + (Q_1 \cos 2ny + Q_2)\sin 2(mx-t) \quad \text{at} \quad z=0, \tag{27}$$

in which

$$\left.\begin{aligned} Q_1 &= \tfrac{1}{4}[3\omega_0^3 - \omega_0^{-1}(2m^2+2n^2+1)], \\ Q_2 &= \tfrac{1}{4}[3\omega_0^3 - \omega_0^{-1}(2m^2-2n^2+1)]. \end{aligned}\right\} \tag{28}$$

A general solution for ϕ_1 which satisfies both Laplace's equation and the other boundary conditions is assumed:

$$\phi_1 = \sum_{r=0}^{\infty} (A_{1r} \cos rmx + B_{1r} \sin rmx) \cos rny \cosh r(z+d)$$
$$+ \sum_{r=0}^{\infty} (C_{1r} \cos rmx + D_{1r} \sin rmx) \cosh rm(z+d), \tag{29}$$

in which A_{1r}, B_{1r}, C_{1r} and D_{1r} are periodic functions of time. Inserting (29) into (27) yields the following differential equations:

$$A_{10tt} + C_{10tt} = 0, \tag{30}$$

$$\begin{Bmatrix} A_{11tt} \\ B_{11tt} \end{Bmatrix} + \begin{Bmatrix} A_{11} \\ B_{11} \end{Bmatrix} = \frac{2\omega_1}{\omega_0^2 \cosh d} \begin{Bmatrix} -\sin t \\ \cos t \end{Bmatrix}, \tag{31}$$

$$\begin{Bmatrix} C_{11tt} \\ D_{11tt} \end{Bmatrix} + \frac{m \tanh md}{\omega_0^2} \begin{Bmatrix} C_{11} \\ D_{11} \end{Bmatrix} = 0, \tag{32}$$

$$\begin{Bmatrix} A_{12tt} \\ B_{12tt} \end{Bmatrix} + \frac{2 \tanh 2d}{\omega_0^2} \begin{Bmatrix} A_{12} \\ B_{12} \end{Bmatrix} = \frac{Q_1}{\omega_0^2 \cosh 2d} \begin{Bmatrix} -\sin 2t \\ \cos 2t \end{Bmatrix}, \tag{33}$$

$$\begin{Bmatrix} C_{12tt} \\ D_{12tt} \end{Bmatrix} + \frac{2m \tanh 2md}{\omega_0^2} \begin{Bmatrix} C_{12} \\ D_{12} \end{Bmatrix} = \frac{Q_2}{\omega_0^2 \cosh 2md} \begin{Bmatrix} -\sin 2t \\ \cos 2t \end{Bmatrix}. \tag{34}$$

From (30)–(34), the coefficients A_{1r}, B_{1r}, C_{1r} and D_{1r} can be determined. For $r \geqslant 3$, the A_{1r}, B_{1r}, C_{1r} and D_{1r} must vanish owing to their periodicity. The resulting solution of (30) is taken as

$$A_{10} + C_{10} = (\alpha_{10} t) + \beta_{10}. \tag{35}$$

The periodicity of A_{11} and B_{11} in (31) requires $\omega_1 = 0$. It may thus be assumed that

$$\{A_{11}, B_{11}, C_{11}, D_{11}\} = (q_i \cos t + \lambda_i \sin t) \quad \text{for} \quad i = 1, \ldots, 4. \tag{36}$$

C_{11} and D_{11} are zero because the coefficient of the second term in (32) is generally not an integer, except for normal progressive waves. For the progressive wave the solutions for C_{11} and D_{11} are included in (36) for $i = 3$ and 4 since the coefficient is unity.

From (33) and (34)

$$\begin{Bmatrix} A_{12} \\ B_{12} \end{Bmatrix} = \frac{(1+\omega_0^4)\,[3\omega_0^{-3} - \omega_0^{-7}(2m^2+2n^2+1)]}{16\cosh 2d} \begin{Bmatrix} \sin 2t \\ -\cos 2t \end{Bmatrix}, \tag{37}$$

$$\begin{Bmatrix} C_{12} \\ D_{12} \end{Bmatrix} = \frac{(1+\omega_m^4)\,[3\omega_0 - \omega_0^{-3}(2m^2-2n^2+1)]}{16\cosh 2md[(1+\omega_m^4) - m(\omega_m/\omega_0)^2]} \begin{Bmatrix} \sin 2t \\ -\cos 2t \end{Bmatrix}, \tag{38}$$

in which $\omega_m^2 = \tanh md$. The values of α_{10} and β_{10} in (35) and the values of q_i and λ_i in (36) must be determined.

Inserting the above results into (29) and substituting ϕ_1 into (25) yields

$$\begin{aligned} \eta_1 = {} & \tfrac{1}{8}(\omega_0^2 - \omega_0^{-2}) - \omega_0\,\alpha_{10} + \omega_0[(q_1 \sin t - \lambda_1 \cos t)\cos mx \\ & + (q_2 \sin t - \lambda_2 \cos t)\sin mx]\cos ny \cosh d + \omega_0[(q_3 \sin t - \lambda_3 \cos t)\cos mx \\ & + (q_4 \sin t - \lambda_4 \cos t)\sin mx]\cosh md \\ & + [\text{other second-order terms in } \cos 2ny \text{ and } \cos 2(mx-t) \text{ in (25)}]. \end{aligned} \tag{39}$$

By applying (13) to η_1 as given by (39), it is found that $\alpha_{10} = \frac{1}{8}(-\omega_0^{-3} + \omega_0)$.

Employing the additional conditions (13)–(18) arising from the limiting cases shows that q_i and λ_i are zero for all i. Thus all the integral constants except β_{10}, which is of no consequence, have now been evaluated. The complete solutions for the second-order approximation can now be written as

$$\phi_1 = \beta_1 t + \beta_2 \cosh 2(z+d)\cos 2ny \sin 2(mx-t) + \beta_3 \cosh 2m(z+d)\sin 2(mx-t), \tag{40}$$

$$\eta_1 = b_1 \cos 2ny \cos 2(mx-t) + b_2 \cos 2(mx-t) + b_3 \cos 2ny, \tag{41}$$

$$\omega_1 = 0, \tag{42}$$

in which

$$\left.\begin{aligned} \beta_1 &= \tfrac{1}{8}(-\omega_0^{-3} + \omega_0), \\ \beta_2 = \frac{3(\omega_0^{-7} - \omega_0)}{16\cosh 2d}, \quad \beta_3 &= \frac{K_2}{16\cosh 2md} = \frac{(1+\omega_m^4)K_1}{16\cosh 2md} \end{aligned}\right\} \tag{43}$$

and

$$\left.\begin{aligned} & b_1 = \tfrac{1}{8}(3\omega_0^{-6} - \omega_0^{-2}), \quad b_2 = \tfrac{1}{8}[3\omega_0^2 - \omega_0^{-2}(m^2-n^2) + \omega_0 K_2], \\ & b_3 = \tfrac{1}{8}[\omega_0^2 - \omega_0^{-2}(m^2-n^2)], \end{aligned}\right\} \tag{44}$$

where

$$\left.\begin{aligned} K_2 &= \frac{(1+\omega_m^4)\,[(2m^2-2n^2+1)\,\omega_0^{-3} - 3\omega_0]}{[(1+\omega_m^4) - m(\omega_m/\omega_0)^2]}, \\ K_1 &= K_2/(1+\omega_m^4), \quad \omega_m^2 = \tanh md. \end{aligned}\right\} \tag{45}$$

It is interesting to observe that ϕ_1 contains one time-dependent term and one progressive form propagating in the x direction, but not in the y direction. Also, η_1 has a non-constant term which is independent of time but dependent upon distance from the wall in the y direction. These are similar to the findings of Tadjbakhsh & Keller (1960). The limiting forms of Stokes waves for $m = 1$ and $n = 0$ and standing waves for $m = 0$ and $n = 1$ can be readily obtained.

Third-order approximation

By collecting terms in ϵ^2, the governing equations in dimensionless form are

$$\phi_{2xx}+\phi_{2yy}+\phi_{2zz} = 0, \tag{46a}$$

$$\tfrac{1}{2}(\eta_2+\omega_0\phi_{2t}) = -\tfrac{1}{2}\omega_2\phi_{0t}-\omega_0(\eta_1\phi_{0zt}+\eta_0\phi_{1zt}+\tfrac{1}{2}\eta_0^2\phi_{0zzt})-\eta_0(\phi_{0x}\phi_{0xz} + \phi_{0y}\phi_{0yz}+\phi_{0z}\phi_{0zz})-\phi_{0x}\phi_{1x}-\phi_{0y}\phi_{1y}-\phi_{0z}\phi_{1z} \quad \text{at} \quad z=0, \tag{46b}$$

$$\tfrac{1}{2}(\phi_{2z}-\omega_0\eta_{2t}) = \tfrac{1}{2}\omega_2\eta_{0t}-(\eta_1\phi_{0zz}+\eta_0\phi_{1zz}+\tfrac{1}{2}\eta_0^2\phi_{0zzz})+\eta_0(\eta_{0x}\phi_{0xz} + \eta_{0y}\phi_{0yz})+\eta_{0x}\phi_{1x}+\eta_{0y}\phi_{1y}+\eta_{1x}\phi_{0x}+\eta_{1y}\phi_{0y} \quad \text{at} \quad z=0, \tag{46c}$$

$$\phi_{2z} = 0 \quad \text{at} \quad z=-d; \quad \phi_{2y}=0 \quad \text{at} \quad y=0. \tag{46d}$$

To find the current unknowns ϕ_2, η_2 and ω_2, the same procedure is followed as in the second-order approximation. Upon eliminating η_2 from (46*b*, *c*), the combined free-surface boundary condition is obtained as

$$\phi_{2z}+\omega_0^2\phi_{2tt} = \alpha_{11}\cos ny\sin(mx-t)+\alpha_{13}\cos 3ny\sin(mx-t) + \alpha_{31}\cos ny\sin 3(mx-t)+\alpha_{33}\cos 3ny\sin 3(mx-t) \quad \text{at} \quad z=0, \tag{47}$$

in which

$$\alpha_{11} = 2\omega_2+\tfrac{1}{16}(-6\omega_0^{-7}+8\omega_0^{-3}+6\omega_0+8\omega_0^5)+\tfrac{1}{4}m\omega_0^2\omega_m^2K_1+\tfrac{1}{8}(\omega_0^4-4m^2+1)K_2 + m^2[\tfrac{1}{16}(-3\omega_0^{-7}+2\omega_0^{-3}-43\omega_0)+\tfrac{1}{4}(m^2-n^2)\omega_0^{-3}] + n^2[\tfrac{1}{16}(-3\omega_0^{-7}+2\omega_0^{-3}+5\omega_0)-\tfrac{1}{4}(m^2-n^2)\omega_0^{-3}], \tag{48a}$$

$$\alpha_{13} = \tfrac{1}{16}(-3\omega_0^{-7}+8\omega_0^{-3}-3\omega_0+2\omega_0^5)+m^2[\tfrac{1}{16}(-6\omega_0^{-7}+4\omega_0^{-3}-10\omega_0)] + n^2[\tfrac{1}{16}(6\omega_0^{-7}-4\omega_0^{-3}-2\omega_0)+\tfrac{1}{4}(m^2-n^2)\omega_0^{-3}], \tag{48b}$$

$$\alpha_{31} = \tfrac{1}{16}(-9\omega_0^{-7}+64\omega_0^{-3}-33\omega_0+18\omega_0^5)+\tfrac{9}{4}m\omega_0^2\omega_m^2K_1+\tfrac{1}{8}(3\omega_0^4-8m^2-1)K_2 + m^2[\tfrac{1}{16}(-18\omega_0^{-7}+4\omega_0^{-3}-30\omega_0)+\tfrac{1}{4}(m^2-n^2)\omega_0^{-3}] + n^2[\tfrac{1}{16}(18\omega_0^{-7}-4\omega_0^{-3}+2\omega_0)], \tag{48c}$$

$$\alpha_{33} = \tfrac{1}{16}(-27\omega_0^{-7}+66\omega_0^{-3}-39\omega_0). \tag{48d}$$

In finding the general solution for ϕ_2, it is assumed that only the first and third harmonic functions exist, so that

$$\phi_2 = A_{20}+\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}[A_{2q-1,2s-1}\cos(2q-1)mx+B_{2q-1,2s-1}\sin(2q-1)mx] \times[\cos(2s-1)ny]\{\cosh[(2q-1)^2m^2+(2s-1)^2n^2]^{\frac{1}{2}}(z+d)\}, \tag{49}$$

in which $A_{2q-1,2s-1}$, $B_{2q-1,2s-1}$ and A_{20} are periodic functions of time. The constants attached to x, y and z allow ϕ_2 to satisfy Laplace's equation. It is readily shown that (49) satisfies the bottom and wall boundary conditions.

Inserting (49) into (47) yields the following differential equations:

$$A_{20tt} = 0, \tag{50}$$

$$\begin{Bmatrix}A_{11tt}\\B_{11tt}\end{Bmatrix}+\begin{Bmatrix}A_{11}\\B_{11}\end{Bmatrix} = \frac{\alpha_{11}}{\omega_0^2\cosh d}\begin{Bmatrix}-\sin t\\ \cos t\end{Bmatrix}, \tag{51}$$

$$\begin{Bmatrix}A_{13tt}\\B_{13tt}\end{Bmatrix}+\frac{\gamma_1\tanh\gamma_1 d}{\omega_0^2}\begin{Bmatrix}A_{13}\\B_{13}\end{Bmatrix} = \frac{\alpha_{13}}{\omega_0^2\cosh\gamma_1 d}\begin{Bmatrix}-\sin t\\ \cos t\end{Bmatrix}, \tag{52}$$

$$\begin{Bmatrix} A_{31tt} \\ B_{31tt} \end{Bmatrix} + \frac{\gamma_3 \tanh \gamma_3 d}{\omega_0^2} \begin{Bmatrix} A_{31} \\ B_{31} \end{Bmatrix} = \frac{\alpha_{31}}{\omega_0^2 \cosh \gamma_3 d} \begin{Bmatrix} -\sin 3t \\ \cos 3t \end{Bmatrix}, \tag{53}$$

$$\begin{Bmatrix} A_{33tt} \\ B_{33tt} \end{Bmatrix} + \frac{3 \tanh 3d}{\omega_0^2} \begin{Bmatrix} A_{33} \\ B_{33} \end{Bmatrix} = \frac{\alpha_{33}}{\omega_0^2 \cosh 3d} \begin{Bmatrix} -\sin 3t \\ \cos 3t \end{Bmatrix}, \tag{54}$$

in which

$$\gamma_1 = (m^2 + 9n^2)^{\frac{1}{2}}, \quad \gamma_3 = (9m^2 + n^2)^{\frac{1}{2}}. \tag{55}$$

For $2q-1$, $2s-1 \geqslant 4$, the factors $A_{2q-1,2s-1}$ and $B_{2q-1,2s-1}$ must vanish owing to their periodicity. From (50), this requires $A_{20} = \alpha_{20} t + \beta_{20}$. From (51), the periodicity of A_{11} and B_{11} requires the coefficient α_{11} to vanish. Hence for the short-crested waves ω_2 is obtained from (48) with $\alpha_{11} = 0$:

$$\begin{aligned}\omega_2 = {} & \tfrac{1}{32}(6\omega_0^{-7} - 8\omega_0^{-3} - 6\omega_0 - 8\omega_0^5) - \tfrac{1}{8} m \omega_0^2 \omega_m^2 K_1 - \tfrac{1}{16}(\omega_0^4 - 4m^2 + 1) K_2 \\ & + m^2[\tfrac{1}{32}(3\omega_0^{-7} - 2\omega_0^{-3} + 43\omega_0) - \tfrac{1}{8}(m^2 - n^2)\omega_0^{-3}] \\ & + n^2[\tfrac{1}{32}(3\omega_0^{-7} - 2\omega_0^{-3} - 5\omega_0) + \tfrac{1}{8}(m^2 - n^2)\omega_0^{-3}].\end{aligned} \tag{56}$$

Furthermore, the solutions to (52)–(54) can readily be found.

After inserting all the results into equation (49) for ϕ_2, a solution for η_2 can be found from (46*b*). In addition, by applying (13) to η_2, it is found that $\alpha_{20} = 0$. Finally, ϕ_2 and η_2 are given in dimensionless form by

$$\begin{aligned}\phi_2 = {} & \beta_{13} \cosh \gamma_1(z+d) \cos 3ny \sin(mx - t) + \beta_{31} \cosh \gamma_3(z+d) \cos ny \sin 3(mx-t) \\ & + \beta_{33} \cosh 3(z+d) \cos 3ny \sin 3(mx - t) + \beta_{20},\end{aligned} \tag{57}$$

$$\eta_2 = [b_{11} \cos ny + b_{13} \cos 3ny] \cos(mx - t) + [b_{31} \cos ny + b_{33} \cos 3ny] \cos 3(mx-t), \tag{58}$$

in which β_{20} is an arbitrary but inconsequential constant just as β_{10} was previously. The other constants are given by

$$\begin{aligned}\beta_{13} = {} & \alpha_{13}/(\gamma_1 \sinh \gamma_1 d - \omega_0^2 \cosh \gamma_1 d) \\ = {} & [16 \cosh \gamma_1 d(\gamma_1 \tanh \gamma_1 d - \omega_0^2)]^{-1} [(-3\omega_0^{-7} + 8\omega_0^{-3} - 3\omega_0 + 2\omega_0^5) \\ & + m^2(-6\omega_0^{-7} + 4\omega_0^{-3} - 10\omega_0) + n^2(6\omega_0^{-7} - 4\omega_0^{-3} - 2\omega_0) + 4n^2(m^2 - n^2)\omega_0^{-3}],\end{aligned} \tag{59a}$$

$$\begin{aligned}\beta_{31} = {} & [16 \cosh \gamma_3 d(\gamma_3 \tanh \gamma_3 d - 9\omega_0^2)]^{-1} [(-9\omega_0^{-7} + 64\omega_0^{-3} - 33\omega_0 + 18\omega_0^5) \\ & + 36 m \omega_0^2 \omega_m^2 K_1 + 2(3\omega_0^4 - 8m^2 - 1) K_2 + m^2(-18\omega_0^{-7} + 4\omega_0^{-3} - 30\omega_0) \\ & + 4m^2(m^2 - n^2)\omega_0^{-3} + n^2(18\omega_0^{-7} - 4\omega_0^{-3} + 2\omega_0)],\end{aligned} \tag{59b}$$

$$\beta_{33} = (128 \cosh 3d)^{-1} (1 + 3\omega_0^4)(9\omega_0^{-13} - 22\omega_0^{-9} + 13\omega_0^{-5}) \tag{59c}$$

and

$$\begin{aligned}b_{11} = {} & \tfrac{1}{16}(5\omega_0^{-4} - 4 + 4\omega_0^4) + \tfrac{1}{8} m \omega_0 \omega_m^2 K_1 + \tfrac{1}{16}(\omega_0^3 + 2m^2\omega_0^{-1} - \omega_0^{-1}) K_2 \\ & + \tfrac{1}{32} m^2[(3\omega_0^{-8} - 2\omega_0^{-4} - 1) - 4(m^2 - n^2)\omega_0^{-4}] \\ & + \tfrac{1}{32} n^2[(3\omega_0^{-8} - 2\omega_0^{-4} - 1) + 4(m^2 - n^2)\omega_0^{-4}],\end{aligned} \tag{60a}$$

$$\begin{aligned}b_{13} = {} & \tfrac{1}{16}(9\omega_0^{-4} - 6 + 2\omega_0^4) - \tfrac{1}{16} m^2(3\omega_0^{-8} + 5) + \tfrac{1}{16} n^2(3\omega_0^{-8} + 1) \\ & + [16(\gamma_1 \tanh \gamma_1 d - \omega_0^2)]^{-1} [(-3\omega_0^{-6} + 8\omega_0^{-2} - 3\omega_0^2 + 2\omega_0^6) + m^2(-6\omega_0^{-6} \\ & + 4\omega_0^{-2} - 10\omega_0^2) + n^2(6\omega_0^{-6} - 4\omega_0^{-2} - 2\omega_0^2) + 4n^2(m^2 - n^2)\omega_0^{-2}],\end{aligned} \tag{60b}$$

$$b_{31} = \tfrac{1}{16}(21\omega_0^{-4} - 10 + 6\omega_0^4) + \tfrac{3}{4}m\omega_0\,\omega_m^2\,K_1 - \tfrac{1}{16}m^2(3\omega_0^{-8}+5) + \tfrac{1}{16}n^2(3\omega_0^{-8}+1)$$
$$+\tfrac{1}{8}(\omega_0^3 - m^2\omega_0^{-1})\,K_2$$
$$+[3/16(\gamma_3\tanh\gamma_3 d - 9\omega_0^2)]\,[(-9\omega_0^{-6}+64\omega_0^{-2}-33\omega_0^2+18\omega_0^6)+36m\omega_0^3\,\omega_m^2\,K_1$$
$$+2K_2(3\omega_0^5-8m^2\omega_0-\omega_0)+n^2(18\omega_0^{-6}-4\omega_0^{-2}+2\omega_0^2)$$
$$+m^2(-18\omega_0^{-6}+4\omega_0^{-2}-30\omega_0^2)+4m^2(m^2-n^2)\,\omega_0^{-2}], \qquad (60c)$$

$$b_{33} = \tfrac{1}{16}(-3\omega_0^{-8}+21\omega_0^{-4}-15)+[16(\tanh 3d - 3\omega_0^2)]^{-1}\,[(-27\omega_0^{-6}+66\omega_0^{-2}-39\omega_0^2)]. \qquad (60d)$$

It is now necessary to confirm that this third-order short-crested wave theory can be reduced to the normal Stokes or standing wave. Substituting $m = 0$ and $n = 1$ for normal standing waves into (45) and (55) results in $\omega_m = 0$, $K_1 = K_2 = -(\omega_0^{-3}+3\omega_0)$ and $\gamma_1 = 3$, $\gamma_3 = 1$. The complete set of constants related to ϕ_2 and η_2 is then directly deduced from (59) and (60), and ω_2 from (56). These are identical to those derived by Goda & Kakizaki (1966) and Tadjbakhsh & Keller (1960) for finite water depths, with the following constants:

$$\left.\begin{aligned}\beta_{13} &= [128\cosh 3d]^{-1}\,(1+3\omega_0^4)\,(3\omega_0^{-9}-5\omega_0^{-1}+2\omega_0^3),\\ \beta_{31} &= [128\cosh d]^{-1}\,(-9\omega_0^{-9}-62\omega_0^{-5}+31\omega_0^{-1}),\\ \beta_{33} &= [128\cosh 3d]^{-1}\,(1+3\omega_0^4)\,(9\omega_0^{-13}-22\omega_0^{-9}+13\omega_0^{-5})\end{aligned}\right\} \qquad (61)$$

and

$$\left.\begin{aligned}b_{11} &= \tfrac{1}{32}(3\omega_0^{-8}+6\omega_0^{-4}-5+2\omega_0^4),\\ b_{13} &= \tfrac{3}{128}(9\omega_0^{-8}+27\omega_0^{-4}-15+\omega_0^4+2\omega_0^8),\\ b_{31} &= \tfrac{1}{128}(-3\omega_0^{-8}-18\omega_0^{-4}+5),\\ b_{33} &= \tfrac{3}{128}(9\omega_0^{-12}-3\omega_0^{-8}+3\omega_0^{-4}-1);\end{aligned}\right\} \qquad (62)$$

also

$$\omega_2 = \tfrac{1}{32}(9\omega_0^{-7}-12\omega_0^{-3}-3\omega_0-2\omega_0^5). \qquad (63)$$

Therefore the present short-crested wave theory does provide direct access to pure standing waves of finite amplitude.

However, in reducing (57) and (58) to Stokes waves, some coefficients need to be combined, as both $\cos ny$ and $\cos 3ny$ tend to unity when $n \to 0$, and $m \to 1$ for Stokes waves. Thus β_{31} and β_{33} should be combined for ϕ_2 in (57); also b_{11} should be combined with b_{13}, and b_{31} with b_{33} for η_2 in (58). Similarly, in (47), the terms on the right-hand side become $(\alpha_{11}+\alpha_{13})\sin(x-t)$ and $(\alpha_{31}+\alpha_{33})\sin 3(x-t)$. Also, the periodicity condition requires $\alpha_{11}+\alpha_{13}$ to vanish, thus giving ω_2 for Stokes waves from (56) with the addition of α_{13} in (48). For Stokes waves, $\gamma_1 = 1$ and $\gamma_3 = 3$ from (55), so that the coefficient $(\gamma_1\tanh\gamma_1 d)/\omega_0^2$ in (52) equals unity, thus modifying (52) to

$$\begin{Bmatrix}A_{13tt}\\ B_{13tt}\end{Bmatrix}+\begin{Bmatrix}A_{13}\\ B_{13}\end{Bmatrix} = \frac{\alpha_{13}}{\omega_0^2\cosh d}\begin{Bmatrix}-\sin t\\ \cos t\end{Bmatrix}, \qquad (64)$$

in which the periodicity of A_{13} and B_{13} for Stokes waves requires the coefficient α_{13} to vanish, just as α_{11} does in (51). Therefore $\beta_{13} = 0$ in (59a). Finally, by letting $m = 1$ and $n = 0$ in (56) and (48b), the equation for ω_2 in dimensionless form is

$$\omega_2 = \tfrac{2}{32}(9\omega_0^{-7}-10\omega_0^{-3}+9\omega_0). \qquad (65)$$

Compared with the third-order Stokes wave theory (Skjelbreia 1959; Laitone 1962), the value of ω_2/ω_0 from (65) is identical to the usual expression

$$\frac{8+\cosh 4kd}{8\sinh^4 kd}, \quad \text{or} \quad \frac{9+8\cosh^4 kd - 8\cosh^2 kd}{8\sinh^4 kd}. \tag{66}$$

Finally the forms of ϕ_2 and η_2 for Stokes waves will be deduced.

In (59), it can be shown that the value of β_{31} is $3\beta_{33}$ for Stokes waves. By inserting the sum of β_{31} and β_{33} into (57) and using (20) and (6), the dimensional quantity ϕ_2 results in the same equation as was obtained by Skjelbreia (1959), namely

$$\begin{aligned}\frac{k^2}{\sigma}\phi_2 &= \tfrac{1}{2}(ka)^3\,\omega_0^{-1}(\beta_{31}+\beta_{33})\cosh 3k(z+d)\sin 3(kx-\sigma t)\\ &= \tfrac{1}{8}(ka)^3\,\frac{3(11-2\cosh 2kd)}{64\sinh^7 kd}\cosh 3k(z+d)\sin 3(kx-\sigma t).\end{aligned} \tag{67}$$

In (60), substitution of $m = 1$, $n = 0$, $\gamma_1 = 1$ and $\gamma_3 = 3$ gives

$$b_{31} = 3b_{33} = \tfrac{3}{128}(27\omega_0^{-12} - 9\omega_0^{-8} + 9\omega_0^{-4} - 3), \tag{68}$$

so that the dimensional form of η_2 from the sum of b_{31} and b_{33} becomes

$$\begin{aligned}\frac{\eta_2}{a} &= \tfrac{1}{2}(ka)^2\,(b_{31}+b_{33})\cos 3(kx-\sigma t)\\ &= (ka)^2\,\frac{3(8\cosh^6 kd+1)}{64\sinh^6 kd}\cos 3(kx-\sigma t).\end{aligned} \tag{69}$$

The agreement with the third-order Stokes wave is thus complete.

4. Wave variables

Mathematical expressions for some pertinent physical variables describing short-crested waves can now be obtained. These include the surface elevation, crest height, wave steepness, wave speed, velocity potential, water-particle velocities and wave pressure.

Surface elevation

From the perturbation series (20) and the results of each order of approximation up to the third, values of ϕ, η and ω can be completely formulated. The dimensional surface elevation η is given by

$$\begin{aligned}k\eta(x,y,t) = \epsilon\hat{\eta}(\hat{x},\hat{y},\hat{t}) &= [(\epsilon+\tfrac{1}{2}\epsilon^3 b_{11})\cos Y + \tfrac{1}{2}\epsilon^3 b_{13}\cos 3Y]\cos X\\ &\quad + [\epsilon^2(b_1\cos 2Y + b_2)]\cos 2X + [\epsilon^2 b_3]\cos 2Y\\ &\quad + [\tfrac{1}{2}\epsilon^3(b_{31}\cos Y + b_{33}\cos 3Y)]\cos 3X,\end{aligned} \tag{70}$$

in which $X = m\hat{x} - \hat{t}$, $Y = n\hat{y}$ and all the constants b_i are given by (44), whilst all the b_{ij} are obtained from (60). It is implicit in (70) that the water surface profiles along the centre-lines of the combined crests (at $y/L_y = 0$, $\frac{1}{2}$, 1 etc.) are similar to the progressive wave. The profile is more likely to be that of a cnoidal wave, when considered to the third order. However, the wave amplitudes are a maximum at these locations and vary along the crest length, being a minimum near the locations half-way

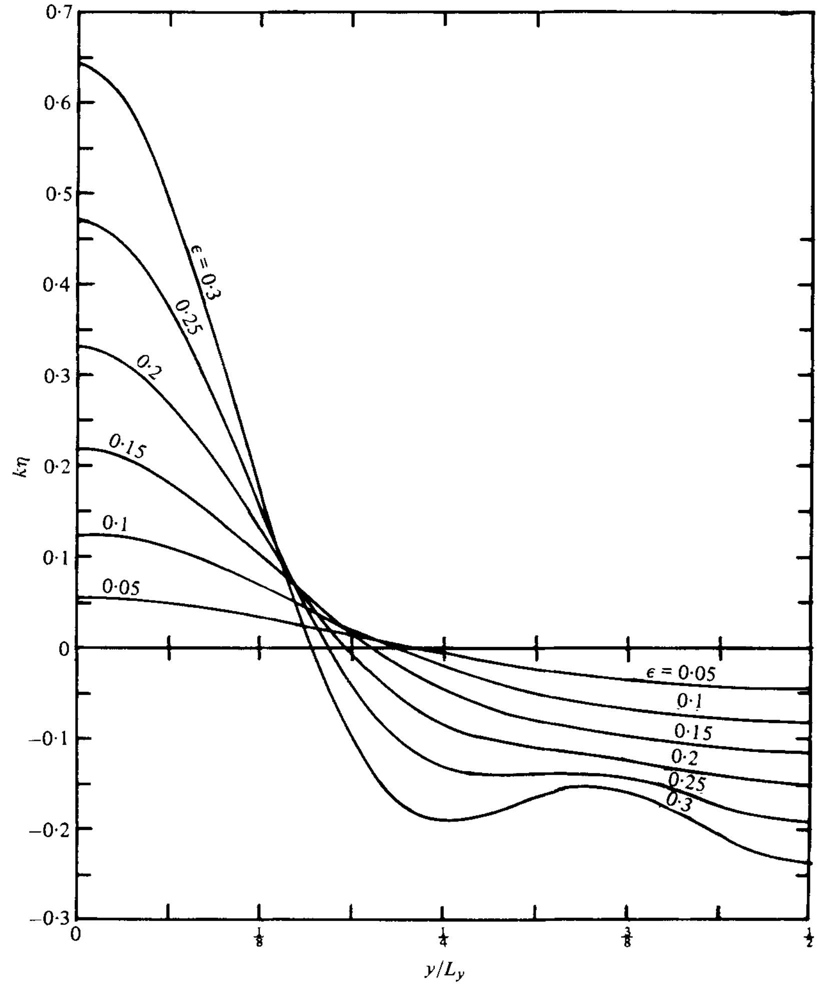

FIGURE 2. Surface profiles of short-crested waves as functions of distance along the crest for different amplitudes in water of constant depth. $\theta = 45°$, $d/L_i = 0{\cdot}1$, $t = 0$.

between (at $y/L_y = \frac{1}{4}, \frac{3}{4}$). Normal to the wall at the crests and troughs, the profile resembles a standing wave.

Examples of the surface profiles are plotted in figures 2 and 3. Figure 2 shows the variation of the surface elevation $k\eta$ according to (70) as a function of distance from the reflecting wall for various dimensionless amplitudes in $\epsilon = ka$ for the case $\theta = \frac{1}{4}\pi$, $d/L_i = 0{\cdot}1$ at time $t = 0$, with L_i the incident wavelength to first order. A comparison of surface profiles for different orders of wave theory is depicted in figure 3. The property of a flat trough and steep crest is more pronounced in the third-order solution.

For the crest along the wall at $x = 0$ and $t = q\pi$ (where q is an even integer), the greatest surface elevation $\eta_{\max}$ in dimensional form as given by (70) is

$$k\eta_{\max} = \epsilon + \epsilon^2(b_1 + b_2 + b_3) + \tfrac{1}{2}\epsilon^3(b_{11} + b_{13} + b_{31} + b_{33}). \tag{71}$$

The lowest section of the trough, $k\eta_{\min}$, occurs at $t = q\pi$, q being an odd integer. Therefore the total wave height H_{sc} of the short-crested wave at the wall is given by

$$kH_{sc} = k[\eta_{\max} - \eta_{\min}]_{x,y=0} = 2\epsilon[1 + \tfrac{1}{2}\epsilon^2(b_{11} + b_{13} + b_{31} + b_{33})]. \tag{72}$$

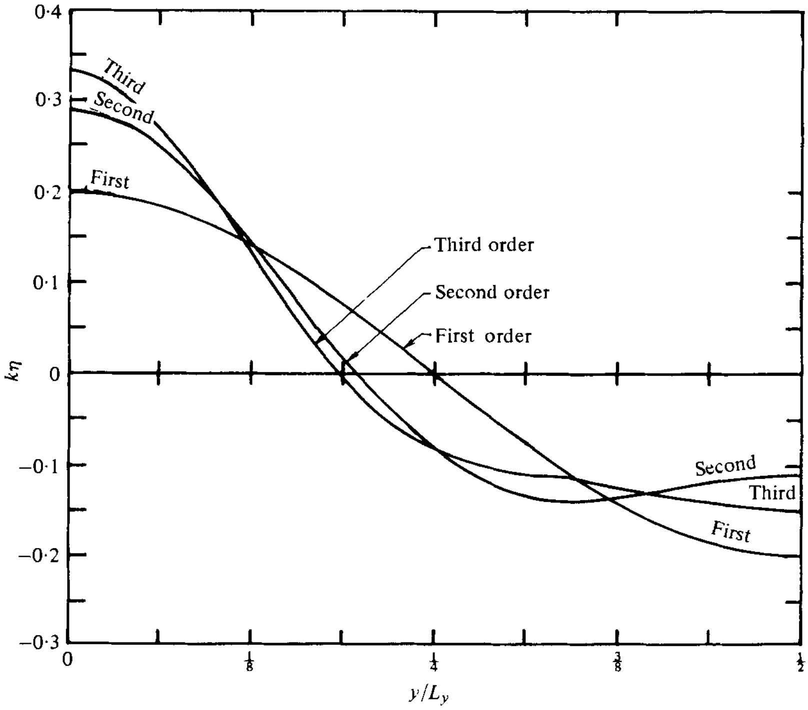

FIGURE 3. Surface profiles of short-crested waves as a function of the order of the wave theory. $\theta = 45°$, $d/L_i = 0{\cdot}1$, $t = 0$, $\epsilon = 0{\cdot}2$.

The presence of b_{11}, b_{13}, b_{31} and b_{33} in (72) influences the total wave height in the short-crested system in the same manner as in Stokes and standing waves to the third order of approximation. By inserting all the constants b_{ij} from (60) into (72), the full expression for H_{sc} can be formulated. It is too lengthy for presentation here. However, the complete expression does lead to the same result as for a normal standing wave, for example

$$kH = \epsilon + \tfrac{1}{256}\epsilon^3(27\omega_0^{-12} + 27\omega_0^{-8} + 96\omega_0^{-4} - 63 + 11\omega_0^4 + 6\omega_0^8), \tag{73}$$

where H is half the wave height to the third order as used by Goda & Kakizaki (1966).

Crest height and wave steepness

The ratio of the maximum crest height $\eta_{\max}$ above the still water level to the total wave height H_{sc} for the short-crested system can be expressed as

$$\frac{\eta_{\max}}{H_{sc}} = \frac{1 + \epsilon(b_1 + b_2 + b_3) + \frac{1}{2}\epsilon^2(b_{11} + b_{13} + b_{31} + b_{33})}{2 + \epsilon^2(b_{11} + b_{13} + b_{31} + b_{33})}. \tag{74}$$

The wave steepness of short-crested waves must account for the change in wavelength of the third-order solution. The perturbed quantity ω from (20) and ω_2 from (56) give

$$\omega = \omega_0 + \tfrac{1}{2}\epsilon^2\omega_2 = \omega_0(1 + \tfrac{1}{2}\epsilon^2\omega_0^{-1}\omega_2), \tag{75}$$

or

$$L = \frac{gT^2}{2\pi}\omega^2 = \frac{gT^2}{2\pi}\omega_0^2[1 + \epsilon^2\omega_0^{-1}\omega_2 + O(\epsilon^3)] = mL_x. \tag{76}$$

This is the same expression as was obtained by Skjelbreia (1959) for a Stokes wave and by Goda & Kakizaki (1966) for a standing wave.

In terms of the distance between combined crests, in the direction of wave propagation, the wavelength L_x provides a steepness ratio from (72) and (76) of

$$\frac{H_{sc}}{L_x} = \frac{m\epsilon[2+\epsilon^2(b_{11}+b_{13}+b_{31}+b_{33})]}{2\pi(1+\epsilon^2\omega_0^{-1}\omega_2)}. \tag{77}$$

Wave speed

The speed of the combined crests in the short-crested system can be transformed by linear wave theory from (23). Since $\cos(m\hat{x}-t) = \cos(m_1 x - \sigma_0 t) = \cos m_1(x - C_s t)$, the wave speed for the first-order solution is defined by

$$C_s = \sigma_0/m_1 = \left[\frac{gT}{2\pi}\tanh kd\right]\bigg/ m_1, \tag{78}$$

in which σ_0 is the angular frequency $2\pi/T$. This is exactly the same as the formula used by Fuchs (1952), with the present notation.

Substitution of the dimensionless angular frequency ω given by (75) in (78) gives the third-order approximation of the wave speed:

$$C_s = \sigma_0(1+\tfrac{1}{2}\epsilon^2\omega_0^{-1}\omega_2)/m_1, \tag{79}$$

in which ω_2 is given by (56).

Velocity potential and water-particle velocities

To non-dimensionalize the water-particle velocities, use is made of the following conditions from variable transforms:

$$\hat{u} = \epsilon(\partial\hat{\phi}/\partial\hat{x}), \quad \hat{v} = \epsilon(\partial\hat{\phi}/\partial\hat{y}), \quad \hat{w} = \epsilon(\partial\hat{\phi}/\partial\hat{z}), \tag{80}$$

in which $\hat{\phi}$ is the perturbed value (20) obtained by inserting the solutions for ϕ_1, ϕ_2 and ϕ_3 from (23), (40) and (57). This leads to the dimensionless velocity potential

$$\begin{aligned}\hat{\phi} = {} & [\omega_0(\cosh Z/\sinh d)\cos Y + \tfrac{1}{2}\epsilon^2\beta_{13}\cosh\gamma_1 Z\cos 3Y]\sin X \\ & + [\epsilon\beta_2\cosh 2Z\cos 2Y + \epsilon\beta_3\cosh 2mZ]\sin 2X \\ & + \tfrac{1}{2}\epsilon^2[\beta_{31}\cosh\gamma_3 Z\cos Y + \beta_{33}\cosh 3Z\cos 3Y]\sin 3X \\ & + \epsilon\beta_1\hat{t} + \beta_{10} + \beta_{20}.\end{aligned} \tag{81}$$

The full forms of the dimensionless particle velocity components in the x, y and z directions can be derived as

$$\begin{aligned}\frac{\hat{u}}{m} = {} & \left(\epsilon\omega_0\frac{\cosh Z}{\sinh d}\cos Y\cos X\right) + \epsilon^2(2\beta_2\cosh 2Z\cos 2Y\cos 2X \\ & + 2\beta_3\cosh 2mZ\cos 2X) \\ & + \tfrac{1}{2}\epsilon^3(\beta_{13}\cosh\gamma_1 Z\cos 3Y\cos X + 3\beta_{31}\cosh\gamma_3 Z\cos Y\cos 3X \\ & + 3\beta_{33}\cosh 3Z\cos 3Y\cos 3X),\end{aligned} \tag{82}$$

$$\begin{aligned}\frac{\hat{v}}{n} = {} & \left(-\epsilon\omega_0\frac{\cosh Z}{\sinh d}\sin Y\sin X\right) - \epsilon^2(2\beta_2\cosh 2Z\sin 2Y\sin 2X) \\ & - \tfrac{1}{2}\epsilon^3(3\beta_{13}\cosh\gamma_1 Z\sin 3Y\sin X + \beta_{31}\cosh\gamma_3 Z\sin Y\sin 3X \\ & + 3\beta_{33}\cosh 3Z\sin 3Y\sin 3X),\end{aligned} \tag{83}$$

$$\hat{w} = \left(\epsilon\omega_0 \frac{\sinh Z}{\sinh d}\cos Y \sin X\right) + \epsilon^2(2\beta_2 \sinh 2Z \cos 2Y \sin 2X$$
$$+ 2m\beta_3 \sinh 2mZ \sin 2X) + \tfrac{1}{2}\epsilon^3(\gamma_1 \beta_{13} \sinh \gamma_1 Z \cos 3Y \sin X$$
$$+ \gamma_3 \beta_{31} \sinh \gamma_3 Z \cos Y \sin 3X + 3\beta_{33} \sinh 3Z \cos 3Y \sin 3X), \tag{84}$$

in which $X = m\hat{x} - \hat{t}$, $Y = n\hat{y}$ and $Z = \hat{z} + \hat{d}$.

Wave pressure

The pressure in the wave motion is given by Bernoulli's theorem in terms of dimensionless quantities as

$$\hat{p} = -z - \epsilon\omega\phi_t - \tfrac{1}{2}\epsilon^2(\phi_x^2 + \phi_y^2 + \phi_z^2), \tag{85}$$

where $\hat{p} = (k/\rho g)p$ is the departure of the wave pressure from that of the atmosphere, ρ being the water density. Omitting the caret as before, the perturbed solution for p is

$$p = -z + \epsilon(p_0 + \epsilon p_1 + \tfrac{1}{2}\epsilon^2 p_2) \tag{86}$$

to the third order of approximation. The substitution of ϕ and ω from (20) into (85) gives the following results for each order of approximation:

$$\left.\begin{aligned} p_0 &= -\omega_0 \phi_{0t}, \quad p_1 = -\omega_0 \phi_{1t} - \tfrac{1}{2}(\phi_{0x}^2 + \phi_{0y}^2 + \phi_{0z}^2), \\ p_2 &= -\omega_2 \phi_{0t} - \omega_0 \phi_{1t} - 2(\phi_{0x}\phi_{1x} + \phi_{0y}\phi_{1y} + \phi_{0z}\phi_{1z}). \end{aligned}\right\} \tag{87}$$

After inserting the solutions for ϕ_0, ϕ_1 and ϕ_2 from (23), (40) and (57) and defining X, Y and Z as above, the dimensionless pressure components are given by

$$p_0 = \frac{\cosh Z}{\cosh d}\cos Y \cos X, \tag{88}$$

$$p_1 = \left[-\omega_0\beta_1 - \frac{\omega_0^2}{8\sinh^2 d}\cosh 2Z\right] + 2\omega_0[\beta_2 \cosh 2Z \cos 2Y + \beta_3 \cosh 2mZ]\cos 2X$$
$$+ \frac{\omega_0^2}{8\sinh^2 d}[n^2 \cosh 2Z - m^2 - \cos 2Y]\cos 2X$$
$$+ \frac{\omega_0^2}{8\sinh^2 d}[n^2 - m^3 \cosh 2Z]\cos 2Y, \tag{89}$$

$$p_2 = \frac{\omega_0}{\sinh d}\{[\omega_2 \cosh Z + \tfrac{1}{2}\beta_2(m-n+1)\cosh Z + \tfrac{1}{2}\beta_2(m-n-1)\cosh 3Z$$
$$+ 2m\beta_3 \cosh(2m-1)Z]\cos Y \cos X + [\tfrac{1}{2}\beta_2(m+n+1)\cosh Z$$
$$+ \tfrac{1}{2}\beta_2(m+n-1)\cosh 3Z + \beta_{13}\sinh d \cosh \gamma_1 Z]\cos 3Y \cos X$$
$$+ [\tfrac{1}{2}\beta_2(m+n-1)\cosh Z + \tfrac{1}{2}\beta_2(m+n+1)\cosh 3Z + 2m\beta_3 \cosh(2m+1)Z$$
$$+ 3\beta_{31}\sinh d \cosh \gamma_3 Z]\cos Y \cos 3X + [\tfrac{1}{2}\beta_2(m-n-1)\cosh Z$$
$$+ \tfrac{1}{2}\beta_2(m-n+1)\cosh 3Z + 3\beta_{33}\sinh d \cosh 3Z]\cos 3Y \cos 3X\}. \tag{90}$$

Appendix

Equation (19) can be written in terms of the dimensionless depth $\hat{d} = 2\pi d/L$ as

$$n' \tanh n'\hat{d}/\tanh \hat{d} \neq j^2 \quad \text{for} \quad n' \geqslant 2, \quad j \geqslant 1. \tag{91}$$

The physical implication of the uniqueness of this equation requires that the frequency

d/L	0·05	0·10	0·153	0·20	0·50
$\hat{d} = 2\pi d/L$	0·314	0·628	0·961	1·257	3·142
$\omega_0^2 = \tanh \hat{d}$	0·304	0·557	0·745	0·850	0·996
$2 \tanh 2\hat{d}/\tanh \hat{d}$	3·661	3·053	2·573	2·321	2·007
$3 \tanh 3\hat{d}/\tanh \hat{d}$	7·262	5·144	4·002	3·525	3·011

TABLE 1. Values of second and third harmonics of (19).

of the nth spatial harmonic is not an integral multiple of the fundamental frequency, which therefore excludes certain fluid depths. Concus (1964) has critically examined this limitation in his paper on standing capillary–gravity waves of finite amplitude. He concluded that it is physically impossible to satisfy this condition in practice as the set of excluded depths is everywhere dense in the interval $(0, \infty)$ for $\hat{d}$, and claimed that this restriction is applicable to the work of Tadjbakhsh & Keller (1960).

Defining ω_0 by $\tanh \hat{d} = \omega_0^2$, as in (23), gives

$$\tanh 2\hat{d} = \frac{2 \tanh \hat{d}}{1 + \tanh^2 \hat{d}} = \frac{2\omega_0^2}{1 + \omega_0^4}$$

and

$$\tanh 3\hat{d} = \frac{3 \tanh \hat{d} + \tanh^3 \hat{d}}{1 + 3 \tanh^2 \hat{d}} = \frac{\omega_0^2(3 + \omega_0^4)}{1 + 3\omega_0^4}.$$

The values of $2 \tanh 2\hat{d}/\tanh \hat{d}$ and $3 \tanh 3\hat{d}/\tanh \hat{d}$ are listed in table 1 for values d/L ranging from shallow to deep water. It is seen that shallow water corresponds to the range from about 3·7 to 2·0 whilst deep water corresponds to the range from 7·3 to 3·0. It is apparent that the excluded depths for the second harmonic occur outside the engineering depth ratios of 0·05 and 0·50.

The third-order harmonic contains a resonance at $d/L = 0{\cdot}153$ since

$$3 \tanh 3\hat{d}/\tanh \hat{d} = 4{\cdot}00.$$

Near resonance can therefore occur over the range $0{\cdot}146 \leqslant d/L \leqslant 0{\cdot}161$ (5% either side). Thus, contrary to the conclusions of Concus (1964), who claimed 'a countable infinity' of excluded depths, these depths appear to be restricted to a narrow band.

For standing waves of finite amplitude, ω_2 in (63) changes sign at $d/L = 0{\cdot}17$ as theoretically predicted by Tadjbakhsh & Keller (1960). Such a frequency reversal effect was subsequently observed experimentally by Fultz (1962), and it was proposed that it occurred at $d/L = 0{\cdot}14$. It is interesting to note that these two values are within 10% either side of $d/L = 0{\cdot}153$. Numerical calculations have indicated that the corresponding d/L value for $\omega_2 = 0$ [see (56)] increases as the approach angle θ of the incident wave increases. The sign change of ω_2 ceases at $\theta \simeq 25°$. It is therefore suggested that the uniqueness condition should be applied for θ less than this value, if it is to be related to ω_2.

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